HARMONIC MORPHISMS AND MOMENT MAPS ON HYPER-KÄHLER MANIFOLDS

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ABSTRACT. We characterise the actions, by holomorphic isometries on a Kähler manifold with zero first Betti number, of an abelian Lie group of $\dim \geq 2$, for which the moment map is horizontally weakly conformal (with respect to some Euclidean structure on the Lie algebra of the group).

Furthermore, we study the hyper-Kähler moment map φ induced by an abelian Lie group T acting by triholomorphic isometries on a hyper-Kähler manifold M, with zero first Betti number, thus obtaining the following:

• If $\dim T = 1$ then φ is a harmonic morphism. Moreover, we illustrate this on the tangent bundle of the complex projective space equipped with the Calabi hyper-Kähler structure, and we obtain an explicit global formula for the map.

• If $\dim T \geq 2$ and either φ has critical points, or M is nonflat and $\dim M = 4 \dim T$ then φ cannot be horizontally weakly conformal.

1. INTRODUCTION

This article is a contribution towards establishing the following statement: *any canonical submersion/projection between Riemannian manifolds is, in a natural way, a (generalized) harmonic morphism*; that is, it preserves, through the pull back, the sheaves of harmonic functions (or natural subsheaves of it).

Devised to produce new harmonic functions on Riemannian manifolds, harmonic morphisms quickly revealed a very strong geometric nature as horizontally weakly conformal harmonic maps, i.e. at the same time a critical point of the energy functional and, away from a critical set, a submersion acting conformally on the horizontal distribution (see [1]).

These combined conditions give an over-determined system, which needs not admit solutions and, indeed, no general existence result has yet emerged. There exist, nonetheless, classification theorems in some instances such as maps from three-dimensional space forms (see [1]) or with one-dimensional fibres on Einstein manifolds [9] , [10] .

A common thread in all these examples is the isometric action of a Lie group on the domain and such favourable conditions can be encountered on a wider scale [8] .

This theme can be taken to an even higher level on Kähler manifolds, since the symplectic structure allows the construction of a moment map which, under conditions, gives a new type of harmonic morphisms.

While, strictly speaking, this scheme fails to produce interesting examples in Kähler geometry, essentially for dimensional reasons (see, also, Theorem 2.4 , and Corollary 2.5 , below), its extension to hyper-Kähler manifolds turns out to be more successful. The existence of a trio of Kählerian structures and, potentially, of triholomorphic isometric actions, allows for moment

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maps which are harmonic morphisms in the three-dimensional Euclidean space (Corollary 3.2). Basically, this constructs three harmonic functions with (mutually) orthogonal gradients.

For higher dimensional abelian Lie group actions, the corresponding hyper-Kähler moment maps are still harmonic and twistorial (Propositions 3.1 and 3.5 , respectively; see [5] for more information on twistorial maps). However, at least in the presence of critical points, or assuming nonflatness and a natural dimensional condition, these cannot be horizontally weakly conformal (Corollary 3.6 , Theorem 3.7 , respectively; see, also, Example 2.2), with respect to the metric induced by some Euclidean structure on the corresponding Lie algebra. Nevertheless (see the proof of Proposition 3.5), the twistoriality is equivalent to the fact that the hyper-Kähler moment map, of any abelian Lie group T acting by triholomorphic isometries, pulls back, into the sheaf of harmonic functions, a natural subsheaf of the sheaf of harmonic functions on $(\mathfrak{t} \otimes \mathrm{Im}\mathbb{H})^*$, where $\mathfrak{t}$ is the Lie algebra of T.

In dimension four, a harmonic morphism is a (local) hyper-Kähler moment map if and only if it is given by the Gibbons–Hawking construction. This leads to natural classes of hyper-Kähler moment maps, for all possible dimensions (Example 3.4).

We, also, implement this approach on the tangent bundle of the complex projective space equipped with Calabi's hyper-Kähler metric, with explicit descriptions of spaces and maps. Working with a natural complex homogeneous model for $\mathbb{C}P^n$, we give detailed formulas for the constituents of the hyper-Kähler structure (see Section 4) and show how Killing fields on $\mathbb{C}P^n$ give rise to Killing fields on $T\mathbb{C}P^n$, with tri-holomorphic isometric actions. Integrating the three Hamiltonian functions provides an explicit globally defined harmonic morphism from $T\mathbb{C}P^n$ to $\mathbb{R}^3$, parametrized by $\mathfrak{su}(n+1)$ (Theorem 5.1). Dimension one must stand out, as for $n = 1$ we obtain a harmonic morphism with one-dimensional fibres and this map must fall into one of the previously described classifications: it is produced, as a projection, by the foliation defined by the distinguished Killing field on $T\mathbb{C}P^1$, in a manner reminiscent of the Hopf map.

2. Horizontally conformal moment maps on Kähler manifolds

All the manifolds are assumed connected. Also, for simplicity, we restrict ourselves to abelian Lie group actions, although most of our results hold in the more general setting of generalized foliations generated by a finite family of commuting (tri)holomorphic isometries.

An orbit of a group action is called *nontrivial* if it is not a fixed point. All the actions are assumed nontrivial (that is, there exists at least one nontrivial orbit).

Proposition 2.1. *Let T be an abelian Lie group acting by holomorphic isometries on a Kähler manifold (M, g, J) with zero first Betti number. Denote by $\varphi : M \to \mathfrak{t}^*$ the moment map, where $\mathfrak{t}$ is the Lie algebra of T.*

The following assertions are equivalent:

(i) *The moment map of T is horizontally weakly conformal, with respect to some Euclidean metric on $\mathfrak{t}$;*

(ii) *There exists an invariant metric on T such that the induced maps from T onto the nontrivial orbits are homothetic covering maps;*

(iii) *The nontrivial orbits of T are umbilical and have the same dimension as T.*

Proof. (i)$\Longleftrightarrow$(ii) If $v \in \mathfrak{t}$ we shall denote by V the corresponding (fundamental) vector field on M.

Let h be a Euclidean metric on $\mathfrak{t}$ and let (v_i) be a basis of $\mathfrak{t}$. Then (i) is satisfied, with respect to h, if and only if (see [1, Lemma 2.4.4]) there exists a nonnegative function Λ on M such that $g(\mathrm{d}\varphi_i, \mathrm{d}\varphi_j) = \Lambda h_{ij}$, for any i and j, where $h_{ij} = h(v_i, v_j)$ and $\varphi_i = v_i \circ \varphi$.

As $\mathrm{d}\varphi_i$ is the one-form which, under the musical isomorphisms, corresponds to JV_i, we have that (i) holds, with respect to h, if and only if there exists a nonnegative function Λ such that $g(V_i, V_j) = \Lambda h_{ij}$, for any i and j; in particular, as T is acting by isometries, Λ is constant along the orbits. The proof of this equivalence quickly follows.

(ii)$\Longleftrightarrow$(iii) A foliation on a Riemannian manifold is umbilical if and only if the local flows of the basic vector fields, restricted to each leaf, are conformal diffeomorphisms. Therefore this equivalence follows from the fact that the metrics induced on the orbits are invariant. $\square$

Here is an application of Proposition 2.1.

Example 2.2. Let $T \subseteq \mathrm{SU}(n+1)$ be an abelian Lie subgroup acting canonically on $\mathbb{C}P^n$ (endowed with the Fubini-Study metric). From Proposition 2.1 we obtain that the corresponding moment map is horizontally weakly conformal if and only if $\dim T = 1$.

Indeed, if $\dim T \geq 2$ then, by the Gauss' equation, the sectional curvature of a plane tangent to a nontrivial orbit would be nonpositive which, together with [1, (11.2.5)] applied to the Hopf projection from S^{2n+1} to $\mathbb{C}P^n$, leads to a contradiction.

The following fact should be well-known but we do not have a reference for it.

Lemma 2.3. *Let T be an abelian Lie group acting by symplectic diffeomorphisms on a symplectic manifold, with zero first Betti number. Denote by $\varphi : M \to \mathfrak{t}^*$ the moment map, where $\mathfrak{t}$ is the Lie algebra of T and by $\mathscr{V}$ the generalized foliation formed by the orbits.*

Then $\ker \mathrm{d}\varphi$ is the orthogonal complement of $\mathscr{V}$ with respect to the symplectic form; in particular, the orbits are isotropic.

Proof. If $v \in \mathfrak{t}$ we shall denote by V the induced vector field on M. By definition, the function $x \mapsto \varphi(x)(v)$ is the Hamiltonian of V; we shall denote it by φ_v.

We have that $\mathrm{d}\varphi(X) = 0$, for some $X \in TM$, if and only if $\mathrm{d}\varphi_v(X) = 0$, for any $v \in \mathfrak{t}$; equivalently, $\omega(V, X) = 0$, for any $v \in \mathfrak{t}$, where ω is the symplectic form.

The fact that $\mathscr{V}$ is isotropic follows from the fact that $\mathscr{V} \subseteq \ker \mathrm{d}\varphi$. $\square$

In the following theorem $T^{\mathbb{C}}$ is any complexification of the abelian Lie group T.

Theorem 2.4. *Let T be an abelian Lie group, $\dim T \geq 2$, acting by holomorphic isometries on a Kähler manifold (M, g, J) with zero first Betti number. Denote by $\varphi : M \to \mathfrak{t}^*$ the moment map, where $\mathfrak{t}$ is the Lie algebra of T.*

Then the following assertions are equivalent:

(i) *φ is horizontally weakly conformal, with respect to some Euclidean metric on $\mathfrak{t}$;*

(ii) *The action of T extends to a holomorphic local action of $T^{\mathbb{C}}$ whose orbits are Kähler submanifolds which, locally and up to homotheties, can be identified with $T^{\mathbb{C}}$ endowed with its canonical complex structure and an invariant Kähler metric.*

Moreover, if (i) *or* (ii) *holds then φ is horizontally homothetic.*

Proof. Denote by $\mathscr{V}$ be the generalized foliation formed by the orbits. From Lemma 2.3, we obtain that $J\mathscr{V}$ is the distribution orthogonal to the fibres of φ. Thus, if (ii) holds then φ is

horizontally homothetic.

To prove (i)$\Longrightarrow$(ii), we may restrict to the open subset of M where φ is a submersion.

As T is abelian and acts by holomorphic diffeomorphisms, $J\mathscr{V}$ and $\mathscr{V}\oplus J\mathscr{V}$ are integrable and T extends to a holomorphic local action of $T^{\mathbb{C}}$ whose orbits are the leaves of $\mathscr{V}\oplus J\mathscr{V}$.

To complete the proof, it is sufficient to consider the case $(M,J)=T^{\mathbb{C}}$.

Let $(v_1,\dots,v_n)$ be an orthonormal basis of $\mathfrak{t}$. Then $(V_1,JV_1,\dots,V_n,JV_n)$ is a commuting frame on $T^{\mathbb{C}}$ which, by Proposition 2.1, is conformal with respect to g; in particular, in an open neighbourhood of each point of $T^{\mathbb{C}}$ there exists coordinates $(s_1,t_1,\dots,s_n,t_n)$ such that $\partial/\partial s_i=V_i$ and $\partial/\partial t_i=JV_i$, for any $i=1,\dots,n$. Then the Kähler form is given by $\omega=\Lambda\sum_i^n \mathrm{d}s_i\wedge\mathrm{d}t_i$, for some positive function Λ. As $\mathrm{d}\omega=0$, Λ must be constant which completes the proof. $\square$

The following result is an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let T be an abelian Lie group acting by holomorphic isometries on a Kähler manifold M with zero first Betti number, $\dim M=2\dim T\geq 4$. Denote by $\varphi:M\to\mathfrak{t}^*$ the moment map, where $\mathfrak{t}$ is the Lie algebra of T.*

If φ is horizontally weakly conformal, with respect to some Euclidean metric on $\mathfrak{t}$, then M is flat.

In all the results of this section we may, obviously, replace the assumption on the first Betti number with the (less restrictive) assumption that the action is Hamiltonian. This, also, applies to the results of the next section.

3. Harmonic morphisms and moment maps on hyper-Kähler manifolds

Let (M,g,J_1,J_2,J_3) be a hyper-Kähler manifold, with zero first Betti number. A Kähler structure J on (M,g) is called *admissible* (for the given hyper-Kähler structure) if there exists $q\in S^2(\subseteq\mathrm{Im}\mathbb{H})$ such that $J=q^jJ_j$.

Proposition 3.1. *Let T be an abelian Lie group acting by triholomorphic isometries on a hyper-Kähler manifold M with zero first Betti number.*

Then the induced moment map, with respect to (the Kähler form of) any admissible Kähler structure on M, is a harmonic map.

Proof. It is sufficient to prove that the Hamiltonian of any triholomorphic Killing vector field, with respect to an admissible Kähler structure, is a harmonic function.

Let (g,J_1,J_2,J_3) be the hyper-Kähler structure of M, and let V be a triholomorphic Killing vector field; denote by ω_1, ω_2, ω_3 the Kähler forms of J_1, J_2, J_3, respectively.

Then $\omega_2+\mathrm{i}\,\omega_3$ is a holomorphic symplectic form on (M,J_1) which is preserved by V and, in particular, by $V^{1,0}$. Consequently, if f_2 and f_3 are the Hamiltonians of V, with respect to ω_2 and ω_3, respectively, then $f_2+\mathrm{i}f_3$ is the (holomorphic) Hamiltonian of $V^{1,0}$ with respect to $\omega_2+\mathrm{i}\,\omega_3$; in particular, $f_2+\mathrm{i}f_3$ is a harmonic function on (M,g). $\square$

Corollary 3.2. *Let V be a triholomorphic Killing vector field on a hyper-Kähler manifold (M,g,J_1,J_2,J_3), with zero first Betti number.*

Then (f_1,f_2,f_3) is a harmonic morphism from (M,g) to $\mathbb{R}^3$, where f_j is the Hamiltonian of V with respect to the Kähler form of J_j, $j=1,2,3$.

Example 3.3. The canonical Poisson structure on the dual of a Lie algebra $\mathfrak{g}$ is characterised by the fact that, for any $A \in \mathfrak{g}$, the corresponding linear function on $\mathfrak{g}^*$ is the Hamiltonian of the vector field determined by the transpose of $\mathrm{ad}A$; in particular, the corresponding symplectic leaves are the co-adjoint orbits.

If $\mathfrak{g}$ is the Lie algebra of a compact semisimple Lie group then (see [2] and the references therein) the complex symplectic structure on any co-adjoint orbit M, on the dual of $\mathfrak{g}^{\mathbb{C}}$, underlies a G-invariant hyper-Kähler structure.

Let J_1 be the complex structure on M, induced from $\mathfrak{g}^{\mathbb{C}}$, and let J_2 and J_3 be admissible Kähler structures such that (J_1, J_2, J_3) satisfy the quaternionic identities.

Then, any $A \in \mathfrak{g}$, defines a complex valued harmonic morphism on M, which gives the last two components of the harmonic morphism of Corollary 3.2, determined by the transpose of $\mathrm{ad}A$.

In contrast, the first component of this harmonic morphism (that is, the Hamiltonian with respect to J_1 of the vector field on M determined by $\mathrm{ad}A$) does not seem to admit such a simple and explicit description (compare [4, Lemma 1]). Nevertheless, in Theorem 5.1, below, we shall give an explicit description for a significant particular case.

Let T be an abelian Lie group acting by triholomorphic isometries on a hyper-Kähler manifold M with zero first Betti number; denote by $\mathfrak{t}$ the Lie algebra of T.

Let $\varphi : M \to (\mathfrak{t} \otimes \mathrm{Im}\mathbb{H})^*$ be the map characterised by the fact that, for any $v \in \mathfrak{t}$ and unit quaternion $q \in \mathrm{Im}\mathbb{H}$, the function $x \mapsto \varphi(x)(v \otimes q)$ is the Hamiltonian of the (fundamental) vector field on M corresponding to v with respect to the admissible Kähler structure corresponding to q.

The map φ is called the *hyper-Kähler moment map*. Note that the codomain of φ is a co-CR quaternionic vector space whose twistor space is $n\mathcal{O}(2)$, where $n = \dim T$ and $\mathcal{O}(2)$ is the holomorphic line bundle, over the Riemann sphere, of Chern number 2 [7].

Example 3.4. Let $a > 0$. Then $(\mathbb{R}^4, g_a)$ is a (self-dual) nonflat complete hyper-Kähler manifold (see [9] and the references therein), where

$$g_a = (a|x|^2 + 1)\, g_0 - \frac{a(a|x|^2 + 2)}{a|x|^2 + 1}(-x_2\, \mathrm{d}x_1 + x_1\, \mathrm{d}x_2 - x_4\, \mathrm{d}x_3 + x_3\, \mathrm{d}x_4)^2 ,$$

with g_0 the canonical Euclidean metric on $\mathbb{R}^4$.

Furthermore, $\varphi : (\mathbb{R}^4, g_a) \to \mathbb{R}^3$, $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\overline{z_2})$, is a harmonic morphism given by the Gibbons-Hawking construction. It follows that φ is the hyper-Kähler moment map of the action of S^1 on $(\mathbb{R}^4, g_a)$ given by $\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta z_2)$.

By taking direct products we easily obtain hyper-Kähler moment maps for all possible dimensions of the abelian Lie group and the hyper-Kähler manifold. Similarly, we obtain harmonic morphisms as in Corollary 3.2 for all possible dimensions of the domain.

Proposition 3.5. *Let T be an abelian Lie group acting by triholomorphic isometries on a hyper-Kähler manifold M with zero first Betti number. Denote by $\varphi : M \to (\mathfrak{t} \otimes \mathrm{Im}\mathbb{H})^*$ the hyper-Kähler moment map, where $\mathfrak{t}$ is the Lie algebra of T.*

Then φ is a twistorial map; equivalently, φ corresponds to a holomorphic map from the twistor space of M to $n\mathcal{O}(2)$, mapping the twistor lines onto the images of sections of $n\mathcal{O}(2)$, where $n = \dim T$.

Proof. From Lemma 2.3 , we obtain that the fixed point set of the action is equal to the zero set S of $\mathrm{d}\varphi$. Hence, $M \setminus S$ is a nonempty open set (as we work only with nontrivial actions).

To prove that φ is twistorial outside S, from [6, Proposition 3.1] it follows that it is sufficient to prove the following fact, for each admissible Kähler structure J on M : *the differential of* φ *maps* $\ker(J+\sqrt{-1})$ *, into the (constant) distribution given by* $\mathfrak{t}^{\mathbb{C}} \otimes p^{\perp}$*, where* $p \in (\mathrm{Im}\mathbb{H}\,)^{\mathbb{C}}$ *(depends only of* J *and) is isotropic with respect to the complexification of the canonical Euclidean structure of* $\mathrm{Im}\mathbb{H}$.

We claim that this property holds with $p = q_1 + \sqrt{-1}\,q_2$, where (q, q_1, q_2) is a positive orthonormal frame on $\mathrm{Im}\mathbb{H}$ with q the unit quaternion corresponding to J. Indeed, let ψ_J : $\mathfrak{t}^* \otimes \mathrm{Im}\mathbb{H} \to \mathfrak{t} \otimes \mathbb{C}$ be the linear projection with kernel $\mathfrak{t}^* \otimes \mathbb{R}q$, where $\mathbb{C} \subseteq \mathrm{Im}\mathbb{H}$ is generated by q_1 and q_2 . On identifying $\mathrm{Im}\mathbb{H} = (\mathrm{Im}\mathbb{H}\,)^*$, through its Euclidean structure, we may compose φ followed by ψ_J thus obtaining a holomorphic map from $(M \setminus S, J)$ to $\mathfrak{t}^{\mathbb{C}}$. The fact that $\psi_J \circ \varphi$ is holomorphic is equivalent to the claim.

Now, obviously, $\psi_J \circ \varphi$ extends to a well-defined holomorphic map on (M, J) ; in particular, the zero set of its differential has empty interior. Consequently, also, the zero set of the differential of φ has empty interior which is sufficient to complete the proof. $\square$

Corollary 3.6. *Let* T *be an abelian Lie group acting by triholomorphic isometries on a hyper-Kähler manifold* M *with zero first Betti number. Denote by* $\varphi : M \to (\mathfrak{t} \otimes \mathrm{Im}\mathbb{H}\,)^*$ *the hyper-Kähler moment map, where* $\mathfrak{t}$ *is the Lie algebra of* T.

If the action has a fixed point then φ *is horizontally weakly conformal, with respect to some Euclidean metric on* $\mathfrak{t}$ *, if and only if* $\dim T = 1$.

Proof. Suppose, towards a contradiction, that $\dim T \geq 2$ and φ is horizontally weakly conformal, with respect to some Euclidean metric on $\mathfrak{t}$. Then, by Theorem 2.4 and Proposition 3.1 , φ is a horizontally homothetic harmonic morphism. Consequently, it is a submersion (see [1, Corollary 4.5.5]); that is, the fixed point set is empty, a contradiction. $\square$

By [8] , any foliation, of codim $\neq 2$, generated by an isometric action of an abelian Lie group is locally defined by harmonic morphisms. In the current setting, more can be said:

Theorem 3.7. *Let* T *be an abelian Lie group acting by triholomorphic isometries on a hyper-Kähler manifold* M *with zero first Betti number,* $\dim M = 4 \dim T \geq 8$. *Let* $\varphi : M \to (\mathfrak{t} \otimes \mathrm{Im}\mathbb{H}\,)^*$ *be the hyper-Kähler moment map, where* $\mathfrak{t}$ *is the Lie algebra of* T.

If φ *is horizontally weakly conformal, with respect to some Euclidean metric on* $\mathfrak{t}$ *, then* M *is flat.*

Proof. By the proof of Corollary 3.6 , φ is a horizontally homothetic submersion. Furthermore, as $\dim M = 4 \dim T$, the orbits are connected components of the fibres of φ . Thus, together with Proposition 2.1 , we obtain that, up to a homothety, φ is a Riemannian submersion. Moreover, Propositions 2.1 and 3.1 imply that the fibres of φ are flat and geodesic.

To complete the proof it is sufficient to prove that the horizontal distribution (that is, the orthogonal complement of $\ker \mathrm{d}\varphi$) is integrable.

If $v \in \mathfrak{t}$ we shall denote by V the induced vector field on M, and if q is an imaginary unit quaternion we shall denote by J_q the corresponding admissible Kähler structure. Note that, for any such v and q , the musical dual of $J_q V$ is the pullback by φ of the differential of the linear function on $(\mathfrak{t} \otimes \mathrm{Im}\mathbb{H}\,)^*$ defined by $v \otimes q$. Equivalently, on identifying $(\mathfrak{t} \otimes \mathrm{Im}\mathbb{H}\,)^* = \mathfrak{t} \otimes \mathrm{Im}\mathbb{H}$

through the isomorphism induced by the Euclidean metrics, we have that J_qV is the horizontal lift of the (constant) vector field on $\mathfrak{t}\otimes\mathrm{Im}\mathbb{H}$ defined by $v\otimes q$.

Consequently, for any imaginary unit quaternions p and q and any vector fields U and V induced by elements of $\mathfrak{t}$, we have that $[J_pU, J_qV]$ is a vertical vector field.

Let p, q and r be orthonormal imaginary quaternions and let U, V and W be vector fields induced by elements of $\mathfrak{t}$. On denoting by ω_p the Kähler form of J_p, we obtain $0 = \mathrm{d}\omega_p(J_pU, J_qV, J_rW) = -g\big(U, [J_qV, J_rW]\big)$, thus completing the proof. $\square$

4. The Calabi metric on the tangent bundle of the complex projective space

Let $\mathrm{HM}(n+1) = \big\{A \in \mathfrak{gl}(n+1,\mathbb{C})\,|\,\bar{A} = A^t\big\}$ be the set of Hermitian $(n+1)\times(n+1)$-matrices. $\mathrm{HM}(n+1)$ is an $(n+1)^2$-dimensional complex vector space and we equip it with the Hermitian metric $(A,B)\mapsto 2\,\mathrm{trace}(AB)$, for A and B in $\mathrm{HM}(n+1)$.

Then (see [11]) $\mathbb{C}P^n$ can be embedded into $\mathrm{HM}(n+1)$ as the subset

$$\big\{A\in\mathrm{HM}(n+1)\,|\,A^2 = A\,,\ \mathrm{trace}\,A = 1\big\}\ .$$

Let $\mathrm{U}(n)$ be the unitary group, then $\mathbb{C}P^n$ is a submanifold of $\mathrm{HM}(n+1)$ diffeomorphic to $\mathrm{U}(n+1)/\big(\mathrm{U}(1)\times\mathrm{U}(n)\big)$. The manifold $\mathbb{C}P^n$ can actually be seen as the orbit of the adjoint action of $\mathrm{U}(n+1)$ on $\mathrm{HM}(n+1)$, at

$$A_0 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\ .$$

With respect to the metric on $\mathrm{HM}(n+1)$, the $\mathrm{U}(n+1)$ action is isometric and the induced metric on $\mathbb{C}P^n$ is isometric with the Fubini–Study metric, with holomorphic sectional curvature equal to 1. Note that, the complex structure J on $\mathbb{C}P^n$ is given by $JX = \mathrm{i}(I-2A)X$, for all $A\in\mathbb{C}P^n$ and $X\in T_A\mathbb{C}P^n$.

Later on, we shall need the following relations which can be checked directly at the point A_0 and then extended to the whole of $\mathbb{C}P^n$ by the $\mathrm{SU}(n+1)$ action (beware that the first one is only valid for $n=1$):

$$\begin{aligned} XY+YX &= \mathrm{trace}(XY)I, \quad X,Y\in T_A\mathbb{C}P^1\ ;\\ (XY+YX)A &= \mathrm{trace}(XY)A, \quad X,Y\in T_A\mathbb{C}P^n\ ;\\ 2(XXY+YXX+2XYX) &= \mathrm{trace}(XX)Y, \quad X,Y\in T_A\mathbb{C}P^n, X\perp Y. \end{aligned} \tag{4.1}$$

Let $\pi: T\mathbb{C}P^n\to\mathbb{C}P^n$ be the projection. As $T\mathbb{C}P^n$ is a vector bundle we have a canonical embedding of $\pi^*\big(T\mathbb{C}P^n\big)$ into $\ker\mathrm{d}\pi$. Accordingly, any $Y\in T_A\mathbb{C}P^n$, for some $A\in\mathbb{C}P^n$, has a *vertical* lift Y^v which is a section of $T\big(T_A\mathbb{C}P^n\big)$ $\big(\subseteq T\big(T\mathbb{C}P^n\big)\big)$.

Under the embedding of $\mathbb{C}P^n$ into $\mathrm{HM}(n+1)$, by using the fact that taking vertical lifts is canonical, we deduce that, for any $X\in T_A\mathbb{C}P^n$ we have $Y^v = (0,Y)\in T_{(A,X)}T\mathbb{C}P^n$.

On the other hand, the Levi–Civita connection of $\mathbb{C}P^n$ determines a splitting of the exact sequence $0\longrightarrow\pi^*\big(T\mathbb{C}P^n\big)\longrightarrow T\big(T\mathbb{C}P^n\big)\longrightarrow\pi^*\big(T\mathbb{C}P^n\big)\longrightarrow 0$, where the projection from $T\big(T\mathbb{C}P^n\big)$ onto $\pi^*\big(T\mathbb{C}P^n\big)$ is induced by $\mathrm{d}\pi$.

Therefore any $Y\in T_A\mathbb{C}P^n$, for some $A\in\mathbb{C}P^n$, also, has a *horizontal* lift $Y^h\in T_{(A,X)}T\mathbb{C}P^n$, for any $X\in T_A\mathbb{C}P^n$. A straightforward argument shows that $Y^h = \big(Y,\,\mathrm{i}\,[X,JY]\big)\in T_{(A,X)}T\mathbb{C}P^n$.

We can now describe *Calabi's hyper-Kähler metric* G on $T\mathbb{C}P^n$ (compare [3]). At each $(A,X)\in T\mathbb{C}P^n$ it is given by

$$
\begin{aligned}
&G(U^h,V^h)=\tfrac{a+1}{2}g(U,V)+\tfrac{a-1}{2|X|^2}\left[g(U,X)g(V,X)+g(U,JX)g(V,JX)\right]\ ;\\
&G(U^h,V^v)=0\ ;\\
&G(U^v,V^v)=\tfrac{2}{a+1}g(U,V)-\tfrac{a-1}{2a(a+1)|X|^2}\left[g(U,X)g(V,X)+g(U,JX)g(V,JX)\right]\ ,
\end{aligned}
$$

where $a=\sqrt{1+4|X|^2}$, $U,V\in T_A\mathbb{C}P^n$ and g is the Fubini-Study metric on $\mathbb{C}P^n$ at the point A.

On $\mathbb{C}P^1$ the formulas greatly simplify into

$$
\begin{aligned}
&G(U^h,V^h)=a\,g(U,V);\\
&G(U^h,V^v)=0;\\
&G(U^v,V^v)=\tfrac{1}{a}\,g(U,V).
\end{aligned}
$$

The three complex structures on $T\mathbb{C}P^n$ are given, at each $(A,X)\in T\mathbb{C}P^n$, as follows:

(1) the canonical complex structure J^* on $T^*\mathbb{C}P^n\ \big(=T\mathbb{C}P^n\big)$:

$$
\begin{aligned}
&J^*(U^h)=(JU)^h\ ,\\
&J^*(U^v)=-(JU)^v\ ;
\end{aligned}
$$

(2) the complex structure I^*:

$$
\begin{aligned}
&I^*(U^h)=\tfrac{a+1}{2}U^v+\tfrac{a-1}{2|X|^2}\left[g(U,X)X+g(U,JX)JX\right]^v\ ,\\
&I^*(U^v)=-\tfrac{2}{a+1}U^h+\tfrac{a-1}{2a(a+1)|X|^2}\left[g(U,X)X+g(U,JX)JX\right]^h\ ;
\end{aligned}
$$

(3) the product $K^*=I^*\circ J^*$:

$$
\begin{aligned}
&K^*(U^h)=\tfrac{a+1}{2}(JU)^v+\tfrac{a-1}{2|X|^2}\left[g(U,X)JX-g(U,JX)X\right]^v\ ,\\
&K^*(U^v)=\tfrac{2}{a+1}(JU)^h-\tfrac{a-1}{2a(a+1)|X|^2}\left[g(U,X)JX-g(U,JX)X\right]^h\ .
\end{aligned}
$$

The Levi–Civita connection $\bar\nabla$ of G is given by

$$
\begin{aligned}
&\bar\nabla_{U^h}V^h=(\nabla_UV)^h-\tfrac12\left[R(U,V)X+R(U,JY)JX\right]^v\ ,\\
&\bar\nabla_{U^v}V^h=\tfrac12\left(\mathcal{A}(U,V)\right)^h\ ,\\
&\bar\nabla_{U^h}V^v=(\nabla_UV)^v+\tfrac12\left(\mathcal{A}(V,U)\right)^h\ ,\\
&\bar\nabla_{U^v}V^v=\left(\mathcal{B}(U,V)\right)^v\ ,
\end{aligned}
$$

where ∇ and R are the Levi–Civita connection and the curvature tensor of $\mathbb{C}P^n$, respectively, and the tensors $\mathcal{A}$ and $\mathcal{B}$ are defined by

$$
\begin{aligned}
\mathcal{A}(U,V)=\ &\tilde a\left[g(U,X)V-g(U,JX)JV\right]\\
&+\tilde a\left[g(U,V)-\tilde a\left[g(U,X)g(V,X)+g(U,JX)g(V,JX)\right]\right]X\\
&+\tilde a\left[g(JU,V)-\tilde a\left[g(U,X)g(V,JX)-g(U,JX)g(V,X)\right]\right]JX
\end{aligned}
$$

and

$$\begin{aligned}\mathcal{B}(U,V) = &-\tfrac{1}{2}\tilde{a}\,[g(V,X)U + g(U,X)V + g(V,JX)JU + g(U,JX)JV] \\ &+ \tfrac{1}{2}\tilde{a}^2\,[g(U,X)g(V,X) - g(U,JX)g(V,JX)]\,X \\ &+ \tfrac{1}{2}\tilde{a}^2\,[g(U,X)g(V,JX) + g(U,JX)g(V,X)]\,JX\end{aligned}$$

where $\tilde{a} = \frac{a-1}{a|X|^2}$ and $\mathcal{B}$ is bilinear, symmetric and satisfies

$$\mathcal{B}(JU,V) = \mathcal{B}(U,JV) = J\mathcal{B}(U,V).$$

One can check that (G, I^*, J^*, K^*) is a hyper-Kähler structure on $T\mathbb{C}P^n$. Moreover, the canonical lift to $T\mathbb{C}P^n$ of the action of $\mathrm{SU}(n+1)$ on $\mathbb{C}P^n$ is isometric and triholomorphic.

5. Harmonic morphisms from $T\mathbb{C}P^n$

In this section, we prove the following result.

Theorem 5.1. *If $u \in \mathfrak{su}(n+1)$, the map $(f_1, f_2, f_3) : T\mathbb{C}P^n \to \mathbb{R}^3$ is a harmonic morphism, where $T\mathbb{C}P^n$ is endowed with the Calabi metric and*

$$\begin{aligned}f_1(A,X) &= g(\mathrm{i}u, JX)\,,\\ f_2(A,X) &= a\,g(A,\mathrm{i}u) - \tfrac{2}{a+1}\,g(X^2,\mathrm{i}u)\,,\\ f_3(A,X) &= g(\mathrm{i}u, X)\,,\end{aligned}$$

with $a = \sqrt{1+4|X|^2}$, for any $(A,X) \in T\mathbb{C}P^n$, and g the Fubini–Study metric.

Proof. Let γ be the Killing field on $\mathbb{C}P^n$, defined by $u \in \mathfrak{su}(n+1)$, and Γ the induced triholomorphic Killing field on $T\mathbb{C}P^n$. By Corollary 3.2, it is sufficient to prove that (f_1, f_2, f_3) is the hyper-Kähler moment map determined by Γ.

Firstly, note that

$$\begin{aligned}&\Gamma : T\mathbb{C}P^n \to T\mathbb{C}P^n\\ &\quad (A,X) \mapsto (\gamma(A), \gamma(X)) = (\gamma(A))^h + (\nabla_X\gamma)^v \in T_{(A,X)}T\mathbb{C}P^n.\end{aligned}$$

Then

$$\begin{aligned}I^*\Gamma(A,X) &= \tfrac{a+1}{2}(\gamma(A))^v + \tfrac{a-1}{2|X|^2}\,[g(\gamma(A),X)X + g(\gamma(A),JX)JX]^v\\ &\quad - \tfrac{2}{a+1}(\nabla_X\gamma)^h + \tfrac{a-1}{2a(a+1)|X|^2}g(\nabla_X\gamma, JX)(JX)^h\,;\\ J^*\Gamma(A,X) &= (J\gamma(A))^h + (-\nabla_X\gamma)^v\,;\\ K^*\Gamma(A,X) &= \tfrac{a+1}{2}(J\gamma(A))^v + \tfrac{a-1}{2|X|^2}\,[g(\gamma(A),X)JX - g(\gamma(A),JX)X]^v\\ &\quad + \tfrac{2}{a+1}(J\nabla_X\gamma)^h + \tfrac{a-1}{2a(a+1)|X|^2}g(\nabla_X\gamma, JX)X^h\,.\end{aligned}$$

Assuming $X \neq 0$, we construct the orthonormal basis $\{E_1 = \frac{X}{|X|}, E_2 = \frac{JX}{|X|}, E_i\}_{i=3,\dots,n}$ of $T_A\mathbb{C}P^n$ and take its horizontal and vertical lifts, with norms

$$\begin{aligned}&|E_1^h|^2 = |E_2^h|^2 = a, |E_i^h|^2 = \tfrac{a+1}{2}, \quad i \geq 3\,;\\ &|E_1^v|^2 = |E_2^v|^2 = \tfrac{1}{a}, |E_i^v|^2 = \tfrac{2}{a+1}, \quad i \geq 3\,,\end{aligned}$$

and recall the formulas for the derivation of a function on the tangent bundle, at the point (A, X):

$$(5.1)\qquad \begin{gathered} Y^v(f\circ\pi) = Y(f)\circ\pi, \quad Y^v(f\circ\pi) = 0, \\ Y^h(f(|X|^2)) = 0, \quad Y^v(f(|X|^2) = 2f'(|X|^2)g(X,Y), \\ Y^h(g(Z,X)\circ\pi) = g(\nabla_Y Z, X)\circ\pi. \end{gathered}$$

1) As $f_1(A,X) = g(\mathrm{i}u, JX)$, we have, since $J\gamma = \mathrm{proj}_{T_A\mathbb{C}P^n}(\mathrm{i}u)$

$$\begin{aligned} E_i^h(f_1) &= E_i^h(g(\mathrm{i}u, JX)) = E_i^h(g(\gamma, X)) \\ &= g(\nabla_{E_i}\gamma, X) = -g(\nabla_X\gamma, E_i), \end{aligned}$$

so

$$\begin{aligned} \sum_{i=1}^n E_i^h(f_1)\frac{E_i^h}{|E_i^h|^2} &= \tfrac{1}{a}\left(-g\left(\nabla_X\gamma, \tfrac{X}{|X|}\right)\tfrac{X^h}{|X|} - g\left(\nabla_X\gamma, \tfrac{JX}{|X|}\right)\tfrac{(JX)^h}{|X|}\right) \\ &\quad - \sum_{i=3}^n \tfrac{2}{a+1} g(\nabla_X\gamma, E_i)E_i^h \\ &= \tfrac{2}{a+1}(-\nabla_X\gamma)^h + \tfrac{a-1}{a(a+1)|X|^2}\left(-g(\nabla_X\gamma, X)X^h - g(\nabla_X\gamma, JX)(JX)^h\right) \end{aligned}$$

which is exactly the horizontal part of $I^*\Gamma$.
For the vertical part,

$$\begin{aligned} E_i^v(f_1) &= E_i^v(g(\mathrm{i}u, JX)) = E_i^v(g(\gamma, X)) \\ &= (0, E_i)(g(\gamma, X)) = g(\gamma, E_i) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^n E_i^v(f_1) &= a\left[g\left(\gamma, \tfrac{X}{|X|}\right)\tfrac{X^v}{|X|} + g\left(\gamma, \tfrac{JX}{|X|}\right)\tfrac{(JX)^v}{|X|}\right] + \tfrac{a+1}{2}\sum_{i=3}^n g(\gamma, E_i)E_i^v \\ &= \tfrac{a+1}{2}\gamma^v + \tfrac{a-1}{2|X|^2}\left[g(\gamma, X)X^v + g(\gamma, JX)(JX)^v\right], \end{aligned}$$

which is the vertical part of $I^*\Gamma$.

2) As $f_3(A,X) = g(\mathrm{i}u, X) = g(J\gamma, X)$, we have

$$E_i^h(f_3) = E_i^h(g(J\gamma, X)) = g(\nabla_{E_i}J\gamma, X) = g(\nabla_X J\gamma, E_i)\ ,$$

so

$$\begin{aligned} \sum_{i=1}^n E_i^h(f_3)\frac{E_i^h}{|E_i^h|^2} &= \tfrac{1}{a}\left[g\left(\nabla_X J\gamma, \tfrac{X}{|X|}\right)\tfrac{X^h}{|X|} + g\left(\nabla_X J\gamma, \tfrac{JX}{|X|}\right)\tfrac{(JX)^h}{|X|}\right] \\ &\quad + \sum_{i=3}^n \tfrac{2}{a+1} g(\nabla_X J\gamma, E_i)E_i^h \\ &= \tfrac{2}{a+1}(\nabla_X J\gamma)^h - \tfrac{a-1}{a(a+1)|X|^2} g(\nabla_X J\gamma, X)X^h, \end{aligned}$$

which is exactly the horizontal part of $K^*\Gamma$.
For the vertical part,

$$\begin{aligned} E_i^v(f_3) &= E_i^v(g(\mathrm{i}u, X)) = E_i^v(g(J\gamma, X)) \\ &= (0, E_i)(g(J\gamma, X)) = g(J\gamma, E_i), \end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} E_i^v(f_3) &= a\left[g\left(J\gamma, \tfrac{X}{|X|}\right)\tfrac{X^v}{|X|} + g\left(J\gamma, \tfrac{JX}{|X|}\right)\tfrac{(JX)^v}{|X|}\right] + \tfrac{a+1}{2}\sum_{i=3}^{n} g(J\gamma, E_i)E_i^v \\ &= \tfrac{a+1}{2}(J\gamma)^v + \tfrac{a-1}{2|X|^2}\left[g(J\gamma, X)X^v + g(\gamma, X)(JX)^v\right]\end{aligned}$$

which is the vertical part of $K^*\Gamma$.

3) Lastly, as $f_2(A,X) = ag(A, \mathrm{i}u) - \frac{2}{a+1}g(XX, \mathrm{i}u)$ and $J^*\Gamma(A,X) = (J\gamma(A))^h + (-J\nabla_X\gamma(A))^v$, we have

$$\mathrm{grad}^{T\mathbb{C}P^n} f_2 = \sum_{i=1}^{n} E_i^h(f_2)\frac{E_i^h}{|E_i^h|^2} + E_i^v(f_2)\frac{E_i^v}{|E_i^v|^2}.$$

Since $X^h = (X, \mathrm{i}[X, JX]) = (X, 2(XAX - AXX))$,

$$\begin{aligned}E_1^h(f_2) &= \tfrac{1}{|X|}X^h(f_2) = \tfrac{1}{|X|}\Big\{aX^h(g(A, \mathrm{i}u)) - \tfrac{2}{a+1}X^h(g(XX, \mathrm{i}u))\Big\} \\ &= \tfrac{1}{|X|}\Big\{ag(A, \mathrm{i}u) - \tfrac{2}{a+1}g(XT + TX, \mathrm{i}u)\Big\},\end{aligned}$$

where $T = 2(XAX - AXX)$ and therefore $XT + TX = 0$, so

$$E_1^h(f_2) = \tfrac{a}{|X|}g(A, \mathrm{i}u).$$

Since $(JX)^h = (JX, 0)$,

$$\begin{aligned}E_2^h(f_2) &= \tfrac{1}{|X|}(JX)^h(f_2) = \tfrac{1}{|X|}\Big\{a(JX)^h(g(A, \mathrm{i}u)) - \tfrac{2}{a+1}(JX)^h(g(XX, \mathrm{i}u))\Big\} \\ &= \tfrac{a}{|X|}g(JX, \mathrm{i}u).\end{aligned}$$

For $i \geq 3$, $(E_i)^h = (E_i, \mathrm{i}[X, JE_i])$ but

$$\mathrm{i}[X, JE_i] = E_iX + XE_i - 2(E_iX + XE_i)A = E_iX + XE_i,$$

because $(E_iX + XE_i)A = \mathrm{trace}(E_iX)A = 0$ (by (4.1)), so $(E_i)^h = (E_i, E_iX + XE_i) = (E_i, T_i)$. Therefore

$$\begin{aligned}E_i^h(f_2) &= aE_i^h(g(A, \mathrm{i}u)) - \tfrac{2}{a+1}E_i^h(g(XX, \mathrm{i}u)) \\ &= ag(E_i, \mathrm{i}u) - \tfrac{2}{a+1}g(T_iX + XT_i, \mathrm{i}u)\end{aligned}$$

but

$$T_iX + XT_i = (E_iX + XE_i)X + X(E_iX + XE_i) = E_iXX + XXE_i - 2XE_iX = \left(\tfrac{\mathrm{trace}\, X^2}{2}\right)E_i,$$

by Equation (4.1), and

$$\begin{aligned}E_i^h(f_2) &= ag(E_i, \mathrm{i}u) - \tfrac{2}{a+1}\left(\tfrac{\mathrm{trace}\, X^2}{2}\right)g(E_i, \mathrm{i}u) \\ &= \left(a - \tfrac{\mathrm{trace}\, X^2}{a+1}\right)g(E_i, \mathrm{i}u).\end{aligned}$$

For the vertical parts,

$$\begin{aligned}E_1^v(f_2) &= \tfrac{1}{|X|}X^v(f_2) = \tfrac{1}{|X|}\Big\{\tfrac{4}{a}|X|^2g(A, \mathrm{i}u) + \tfrac{2c}{a(a+1)}|X|^2g(XX, \mathrm{i}u) - \tfrac{2}{a+1}X^v(g(XX, \mathrm{i}u))\Big\} \\ &= \tfrac{1}{|X|}\Big\{\tfrac{4}{a}|X|^2g(A, \mathrm{i}u) - \tfrac{2}{a}g(XX, \mathrm{i}u)\Big\},\end{aligned}$$

since $X^v(g(XX,\mathrm{i}u))=X(g(XX,\mathrm{i}u))=2g(XX,\mathrm{i}u)$. However

$$\begin{aligned}E_2^v(f_2)&=\tfrac{1}{|X|}(JX)^v(f_2)=\tfrac{1}{|X|}\Big\{-\tfrac{2}{a+1}g(X(JX)+(JX)X,\mathrm{i}u)\Big\}\\&=0,\end{aligned}$$

by (5.1) and (4.1).
Finally, for similar reasons, if $i\geq 3$

$$E_i^v(f_2)=-\tfrac{2}{a+1}g(XE_i+E_iX,\mathrm{i}u),$$

hence

$$\begin{aligned}\mathrm{grad}^{T\mathbb{C}P^n}f_2=&\tfrac{1}{|X|^2}\Big\{g(X,\mathrm{i}u)X^h+g(JX,\mathrm{i}u)(JX)^h\Big\}+\tfrac{2}{a+1}\Big(a-\tfrac{\mathrm{trace}\,X^2}{a+1}\Big)g(E_i,\mathrm{i}u)E_i^h\\&+\Big\{4g(A,\mathrm{i}u)-\tfrac{2}{|X|^2}g(XX,\mathrm{i}u)\Big\}X^v-g(XE_i+E_iX,\mathrm{i}u)E_i^v.\end{aligned}$$

To establish equality with $J^*\Gamma$, we compute

$$-J\nabla_X\gamma=g(-J\nabla_X\gamma,X)\frac{X}{|X|^2}+g(-J\nabla_X\gamma,JX)\frac{JX}{|X|^2}+g(-J\nabla_X\gamma,E_i)E_i,$$

and

$$\begin{aligned}g(-J\nabla_X\gamma,E_i)&=g(\nabla_X\gamma,JE_i)=g(D_X\gamma,JE_i)=g(\gamma(X),JE_i)\\&=g([u,X],JE_i)=g([X,JE_i],u)\end{aligned}$$

with

$$\begin{aligned}[X,JE_i]&=\mathrm{i}[X,[E_i,A]]=\mathrm{i}\{XE_iA-XAE_i+AE_iX-E_iAX\}\\&=\mathrm{i}\{(E_iX+XE_i)A-XAE_i-E_iAX\}=\mathrm{i}\{2(E_iX+XE_i)A-(XE_i+E_iX)\}\\&=-\mathrm{i}(XE_i+E_iX)\end{aligned}$$

since $(E_iX+XE_i)A=\mathrm{trace}(XE_i)A=0$ by (4.1). Therefore

$$g(-J\nabla_X\gamma,E_i)=g(-(XE_i+E_iX),\mathrm{i}u).$$

On the other hand

$$g(-J\nabla_X\gamma,JX)=-g(\nabla_X\gamma,X)=0,$$

and

$$\begin{aligned}g(-J\nabla_X\gamma,X)&=g(\nabla_X\gamma,JX)=g(\gamma(X),JX)=g([u,X],JX)\\&=g([X,JX],u)=2\mathrm{i}g(2XXA-XX,u)=2\mathrm{i}g((\mathrm{trace}\,XX)A-XX,u)\\&=2(\mathrm{trace}\,XX)g(A,\mathrm{i}u)-2g(XX,\mathrm{i}u)\quad\text{(by Formula (4.1))}.\end{aligned}$$

Therefore

$$\Big(-J\nabla_X\gamma\Big)^v=\frac{1}{|X|^2}\Big\{2(\mathrm{trace}\,XX)g(A,\mathrm{i}u)-2g(XX,\mathrm{i}u)\Big\}X^v-g(XE_i+E_iX,\mathrm{i}u)E_i^v$$

For the horizontal part

$$\begin{aligned}J\gamma(A)&=\mathrm{i}(I-2A)[u,A]=\mathrm{i}\Big(uA-Au-2A(uA-Au)\Big)\\&=\mathrm{i}\Big(uA+Au-2AuA\Big)=(\mathrm{i}u)^{T_A\mathbb{C}P^n},\end{aligned}$$

so

$$J\gamma(A) = g(\mathrm{i}u, X)\frac{X^h}{|X|^2} + g(\mathrm{i}u, JX)\frac{(JX)^h}{|X|^2} + g(\mathrm{i}u, E_i)E_i^h,$$

which is indeed equal to the horizontal part of $\mathrm{grad}^{T\mathbb{C}P^n} f_2$ since

$$\tfrac{2}{a+1}\left(a - \tfrac{\mathrm{trace}\, XX}{a+1}\right) = 1.$$

□

Remark 5.2. It is known that on a $2n$-dimensional compact Kähler-Einstein manifold with positive scalar curvature s, a vector field is Killing if and only it is equal to J times the gradient field of an eigenfunction of the Laplacian for the eigenvalue s/n. While $T\mathbb{C}P^n$ is not compact this property persists for Killing fields which naturally come from Killing fields on $\mathbb{C}P^n$. Contrastingly, the Killing field $(A, X) \mapsto (JX)^v$ satisfies this property for J^* but not for I^*.

Remark 5.3. On $S^2 = \mathbb{C}P^1$, the expressions for the functions simplify to

$$\begin{aligned} f_1(p, e) &= g(b, Je); \\ f_2(p, e) &= \sqrt{1 + |e|^2} g(b, p); \\ f_3(p, e) &= g(b, e), \end{aligned}$$

where $(p, e) \in T\mathbb{S}^2$ and the Killing field γ on $\mathbb{S}^2$ is defined by rotations around $b \in \mathbb{R}^3$.

This gives a new description for [9, Example 4.7], in the case when the starting harmonic function has exactly two singular points.

References

[1] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.), no. 29, Oxford Univ. Press, Oxford, 2003.

[2] O. Biquard, Sur les équations de Nahm et la structure de Poisson des algèbres de Lie semi-simples complexes, *Math. Ann.*, **304** (1996) 253–276.

[3] O. Biquard, P. Gauduchon, Hyper-Kähler metrics on cotangent bundles of Hermitian symmetric spaces, *Geometry and physics (Aarhus, 1995)*, Lecture Notes in Pure and Appl., 184 (1997) 287–298.

[4] S. D'Amorim Santa-Cruz, Twistor geometry for hyper-Kähler metrics on complex adjoint orbits, *Ann. Global Anal. Geom.*, **15** (1997) 361–377.

[5] E. Loubeau, R. Pantilie, Harmonic morphisms between Weyl spaces and twistorial maps II, *Ann. Inst. Fourier (Grenoble)*, **60** (2010) 433–453.

[6] S. Marchiafava, L. Ornea, R. Pantilie, Twistor Theory for CR quaternionic manifolds and related structures, *Monatsh. Math.*, **167** (2012) 531–545.

[7] S. Marchiafava, R. Pantilie, Twistor Theory for co-CR quaternionic manifolds and related structures, *Israel J. Math.*, (in press; DOI: 10.1007/s11856-013-0001-3).

[8] R. Pantilie, Isometric actions and harmonic morphisms, *Michigan Math. J.*, **47** (2000) 453–467.

[9] R. Pantilie, Harmonic morphisms with 1-dim fibres on 4-dim Einstein manifolds, *Comm. Anal. Geom.*, **10** (2002) 779–814.

[10] R. Pantilie, J. C. Wood, Harmonic morphisms with one-dimensional fibres on Einstein manifolds, *Trans. Amer. Math. Soc.*, **354** (2002) 4229–4243.

[11] A. Ros, Spectral geometry of CR-minimal submanifolds in the complex projective space, *Kodai Math. J.*, **6** (1983) 88–99.

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